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# HEAT TRANSFER FROM A HORIZONTAL CIRCULAR WIRE AT SMALL REYNOLDS AND GRASHOF NUMBERS—I

## PURE CONVECTION

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**Abstract**—Heat transfer from a horizontal fine cylinder by pure forced convection at small Reynolds numbers or by pure free convection at small Grashof numbers is analyzed without restriction on Prandtl number by the method of joining the circumferential average temperature in the concentric layer around the cylinder governed mainly by conduction to that in the wake or plume governed mainly by convection. The agreement between analysis and experiment is satisfactory.

### NOMENCLATURE

$a$, radius of the circular wire;
$c_r, c_\theta$, dimensionless velocity components;
$D$, coefficient of the expansion term in the near-field solution;
$g$, acceleration due to gravity;
$Gr$, Grashof number $(= g\beta(T_w - T_\infty)a^3/\nu^2)$;
$h$, average heat-transfer coefficient;
$k$, thermal conductivity;
$Nu$, Nusselt number $(= ha/k)$;
$p$, dimensionless excess pressure;
$Pr$, Prandtl number;
$r, \theta$, dimensionless cylindrical coordinates;
$Re$, Reynolds number $(= U_0 a/\nu)$;
$t$, dimensionless temperature;
$u, v$, dimensionless velocity components;
$x, y$, dimensionless coordinates.

Greek symbols

$\beta$, expansion coefficient;
$\eta$, similarity variable for free convection $[= (Gr\Theta_0)^{1/4} x y^{-2/5}]$;
$\nu$, kinematic viscosity;
$\xi$, similarity variable for forced convection $(= Re^{1/2} x^{-1/2} y)$;
$\rho$, density;
$\Phi$, stream function for forced convection $(\partial\Phi/\partial y = u, \partial\Phi/\partial x = -v)$;
$\Psi$, stream function for free convection $(\partial\Psi/\partial y = -u, \partial\Psi/\partial x = v)$.

Subscripts

0, 1, 2, order of expanded terms;
$j$, at the joining point;
$m$, at the arithmetic mean temperature $= \frac{1}{2}(T_w + T_\infty)$;
$w$, at the surface of the cylinder;
$\infty$, at infinity.

Superscripts

$\bar{\ }$, circumferential average in the near field;
$\hat{\ }$, circumferential average in the far field.

### 1. INTRODUCTION

HEAT transfer from a circular cylinder by pure forced or pure free convection has been widely studied because of its engineering importance. As a fundamental situation of the problem, the present study is concerned with the heat transfer from a horizontal infinite circular cylinder by pure forced convection at small Reynolds numbers or by pure free convection at small Grashof numbers. The convection flow is steady and laminar with its orientation or its direction perpendicular to the cylinder. Here, the above term "small Reynolds number" or "small Grashof number" signifies that the thickness of the boundary layer around the cylinder is sufficiently large compared with its diameter.

For the analysis of the present problem with small characteristic parameter (Reynolds number or Grashof number), the procedure of asymptotic expansion can not readily be employed because the energy equation for the zeroth-order expansion (the conduction equation) has no steady solution maintaining a finite temperature difference between the cylinder surface and infinity. For the case of pure forced convection,

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however, the heat transfer at small Reynolds numbers has been analyzed with the method of matching the solutions expanded asymptotically in the two regions by Wood [1] and by Hieber and Gebhart [2] with the help of the solution for the flow surrounding the cylinder. For the case of pure free convection, the structure of similarity solutions in the upward plume above the cylinder has been investigated in detail by Kuiken and Zeëv Rotem [3]. Undertaking a theoretical investigation for pure free convection at small Grashof numbers, Mahony [4] determined the heat-transfer rates from the cylinder with the method of joining the temperatures of the two solutions. The correlation line of heat transfer which he obtained with joining smoothly the temperature of a solution of the conduction equation around the cylinder to the maximum temperature of a similarity solution in the plume, at a point on the symmetrical axis above the cylinder, however, does not agree well with the experimental results.

Similarly, the temperature-field concerned will be considered to be divided into the two fields as follows. One is "the near field" around the cylinder where conduction is dominant in comparison with convection and the other is "the far field" at a large distance above the cylinder in the plume where convection is dominant, for the case of pure free convection. In order to attempt to obtain the heat-transfer correlation which agrees well with the experimental results at small Grashof numbers, the method of joining smoothly the circumferential average temperatures of the above two fields is chosen from the viewpoint of physical grounds, although the temperatures are not to coincide mathematically owing to the difference of their integral directions. The average temperature solution in the near field may include expanded terms. The far-field similarity solution is calculated without any restriction on Prandtl number to obtain the correlation of the Nusselt number with Grashof number. For the case of pure forced convection, the heat transfer can be similarly analyzed to result in a simple expression of the Nusselt number correlated with Reynolds number. The obtained correlations of the heat transfer in both cases were compared with the experimental results with long fine moving wires in air enclosed in a large box.

## 2. PURE FREE CONVECTION

Consider the heat transfer by pure free convection from an infinite horizontal circular cylinder, of radius $a$, maintained at a uniform temperature $T_w$ to a surrounding infinite body of fluid whose temperature at large distances from the cylinder is $T_\infty$. This phenomena will be presented in a two-dimensional plane and the coordinates system is taken as indicated in Fig. 1.

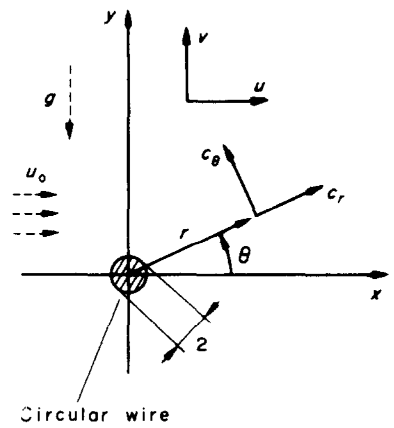



FIG. 1. Coordinates system.

Let $\nu$ be the kinematic viscosity of the fluid, $Pr$ its Prandtl number, $\rho$ its density and $g$ the acceleration due to gravity. It will be assumed that the three quantities $\nu$, $Pr$ and $g$ are effectively constant throughout the fluid and that the fluid density is uniform except as it relates to the buoyancy effect and then taken to be a function of temperature only.

Taking the coordinates, the temperature of the fluid and the velocity components rendered non-dimensional as $(ax, ay)$, $T_\infty + (T_w - T_\infty).t$ and $(\nu u/a,\ \nu v/a)$, respectively, we may write the dimensionless equations of conservation of momentum, mass and energy as

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x}+\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right), \tag{2.1}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y}+\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)+Gr.t, \tag{2.2}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0, \tag{2.3}$$

$$u\frac{\partial t}{\partial x}+v\frac{\partial t}{\partial y} = \frac{1}{Pr}\left(\frac{\partial^2 t}{\partial x^2}+\frac{\partial^2 t}{\partial y^2}\right). \tag{2.4}$$

Here $p$ is the dimensionless excess pressure and $Gr$ the Grashof number, defined by

$$Gr = g\beta(T_w - T_\infty)a^3/\nu^2,$$

where $\beta$ is the coefficient of expansion of the fluid, assumed constant throughout the fluid. The boundary conditions which the solution of the problem must satisfy are

$$\left.\begin{array}{lll} t = 1,\ u = v = 0 & \text{on} & r^2 = x^2+y^2 = 1, \\ t \to 0 & \text{as} & r \to \infty, \end{array}\right\} \tag{2.5}$$

together with suitable behaviour of the velocity field at infinity.

The Nusselt number $Nu$ is defined by

$$Nu = ha/k,$$

where $h$ is the circumferential average of heat-transfer coefficient and $k$ the thermal conductivity of the fluid.

Now, consider the near and far fields around the cylinder separately and then examine to join their solutions.

(i) *The near field*

A near-field solution can be given only with mass and energy equations since conduction is dominant over convection. In order to obtain the circumferential average temperature, equations (2.3) and (2.4) are expressed in the cylindrical coordinates form as follows

$$\frac{\partial(rc_r)}{\partial r}+\frac{\partial c_\theta}{\partial \theta}=0, \tag{2.3}'$$

$$c_r\frac{\partial t}{\partial r}+\frac{c_\theta}{r}\frac{\partial t}{\partial \theta}=\frac{1}{Pr}\left(\frac{\partial^2 t}{\partial r^2}+\frac{1}{r}\frac{\partial t}{\partial r}+\frac{1}{r^2}\frac{\partial^2 t}{\partial \theta^2}\right), \tag{2.4}'$$

where the velocity components are expressed as

$$(vc_r/a,\ vc_\theta/a).$$

Equation (2.4)′ is integrated from zero to $2\pi$ with respect to $\theta$, using equation (2.3)′.

$$Pr\frac{\partial}{\partial r}(\overline{rc_r t})=\frac{\partial}{\partial r}\left(r\frac{\partial \bar{t}}{\partial r}\right), \tag{2.6}$$

where

$$\bar{f}=\frac{1}{2\pi}\int_0^{2\pi} f\,\mathrm{d}\theta$$

means a circumferential average. Further, integration of equation (2.6) from 1 to $r$ with respect to $r$ with the boundary condition (2.5) yields

$$Pr(\overline{rc_r t})=r\frac{\partial \bar{t}}{\partial r}+Nu, \tag{2.7}$$

where $Nu=-(\mathrm{d}\bar{t}/\mathrm{d}r)_{r=1}$ from its definition. Since equations (2.3)′, (2.4)′ and (2.5) give the following relation at the cylinder surface ($r=1$),

$$\overline{rc_r t}=\frac{\partial}{\partial r}(\overline{rc_r t})=\frac{\partial^2}{\partial r^2}(\overline{rc_r t})=0, \tag{2.8}$$

the l.h.s. of equation (2.7) may be expanded in the term of $(r-1)$ to yield,

$$r\frac{\partial \bar{t}}{\partial r}=-Nu+D.(r-1)^3+D_1.(r-1)^4+\ldots. \tag{2.9}$$

If the terms up to the second in the r.h.s. of equation (2.9) are included, equation (2.9) is expressed in the integral form of

$$\bar{t}=1-Nu\ln r+D\int_1^r\frac{1}{r}(r-1)^3\,\mathrm{d}r. \tag{2.10}$$

This equation will be valid for any flow patterns of the far-field solution. If the first term is only used, it leads to the solution of the conduction equation,

$$\bar{t}=1-Nu\ln r. \tag{2.11}$$

The values of $Nu$ and $D$ in equations (2.10) and (2.11) must be determined with the conditions obtained in the far field.

(ii) *The far field*

Even in the range of small Grashof numbers, the upward plume equivalent to the usual boundary layer will exist at a sufficiently large distance above the cylinder since the Grashof number based on its distance becomes large. It is thus considered that the governing equations for the far field are compatible with the existence of the similarity solution. In order to obtain the series of similarity solutions in the plume, a solution of the fundamental equations must be attempted by expanding the stream function for free convection $\Psi$, the temperature $t$ and the excess pressure $p$ in the power series of $y^{-6/5}(\ll 1)$.

$$\Psi=y^{3/5}\{\pi_0.\varphi_0(\eta)+y^{-6/5}\pi_1.\varphi_1(\eta)+(y^{-6/5})^2\pi_2.\varphi_2(\eta)+\ldots\}, \tag{2.12}$$

$$t=y^{-3/5}\{\Theta_0.\theta_0(\eta)+y^{-6/5}\Theta_1.\theta_1(\eta)+(y^{-6/5})^2\Theta_2.\theta_2(\eta)+\ldots\}, \tag{2.13}$$

$$p=y^{-4/5}\{\Gamma_0.\gamma_0(\eta)+y^{-6/5}\Gamma_1.\gamma_1(\eta)+(y^{-6/5})^2\Gamma_2.\gamma_2(\eta)+\ldots\}, \tag{2.14}$$

where $\eta$ is the similarity variable for free convection given by $\eta=Z_G.xy^{-2/5}$.

By considering that the velocity and temperature in the plume are symmetric with respect to the $y$-axis and that the energy passing through an arbitrary cross-section of the plume keeps a constant value given by the heat flux released from the surface of the cylinder, the boundary conditions will be written as follows

$$\left.\begin{aligned} &x=0:\quad \frac{\partial\Psi}{\partial y}=\frac{\partial^2\Psi}{\partial x^2}=\frac{\partial t}{\partial x}=0,\quad \Psi=0;\\ &x\to\infty:\ \frac{\partial\Psi}{\partial x}\to 0,\quad t\to 0;\\ &\int_0^\infty\left(\frac{\partial\Psi}{\partial x}t-\frac{1}{Pr}\frac{\partial t}{\partial y}\right)\mathrm{d}x=\pi\frac{Nu}{Pr}. \end{aligned}\right\} \tag{2.15}$$

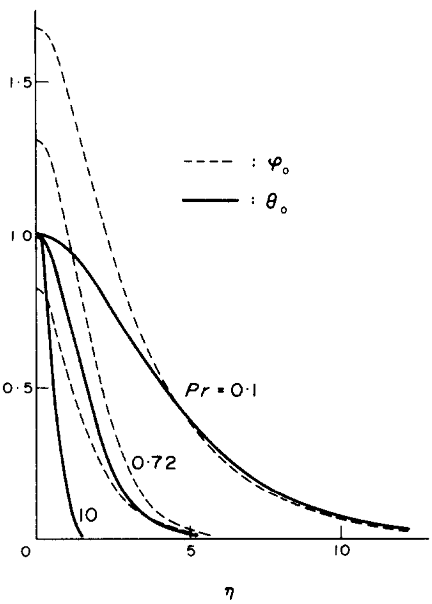
Substituting the above expansions (2.12), (2.13) and (2.14) into equations (2.1), (2.2), (2.4) and (2.15) and equating the coefficients of the like powers of $y^{-6/5}$, we obtain a set of the ordinary differential equations.

Thus, the equations and boundary conditions governing the zeroth-order approximation become

$$-\tfrac{3}{5}\varphi_0\varphi_0'' + \tfrac{1}{5}(\varphi_0')^2 = \varphi_0''' + \theta_0, \qquad (2.16)$$

$$\tfrac{3}{5}\varphi_0\theta_0 = -\frac{1}{Pr}\theta_0', \qquad (2.17)$$

$$\left.\begin{aligned} &\eta = 0: \quad \varphi_0 = \varphi_0'' = 0, \quad \theta_0 = 1; \\ &\eta \to \infty: \ \varphi_0' \to 0; \\ &\Theta_0 = \left\{\frac{1}{\pi} Pr Gr^{1/4} Nu^{-1} \int_0^\infty \varphi_0'\theta_0 \, d\eta\right\}^{-4/5}, \end{aligned}\right\} \qquad (2.18)$$




(a)

where the prime denotes the derivatives with respect to $\eta$. $Z_G = (Gr\Theta_0)^{1/4}$ and $\pi_0 = (Gr\Theta_0)^{1/4}$ are evaluated so as to have $Gr$ eliminated from equations (2.16) and (2.17), and $\Theta_0$ must satisfy the above integral condition. The momentum equation in the $x$-direction need not be considered explicitly because it is only to determine $\gamma_0$ with the solutions of $\varphi_0$ and $\theta_0$. To the boundary condition (2.18), for the convenience of calculation, the condition $\theta_0(0) = 1$ was added.

Here, simultaneous non-linear equations (2.16) and (2.17) subject to the boundary condition (2.18) are solved numerically by the Runge–Kutta–Gill method. The finally convergent solution of $\varphi_0'$ was sought by means of varying an initial value of $\varphi_0'(0)$ so that the corresponding solution might satisfy the boundary conditions as $\eta \to \infty$. The vertical velocity component $\varphi_0'$ and the temperature $\theta_0$ are shown in Fig. 2(a) for $Pr = 0{\cdot}1$, $0{\cdot}72$ and $10$. The vertical velocity to this order approximation tends to infinity like $y^{1/5}$, but it is possible that the flow may actually be turbulent at large distances.

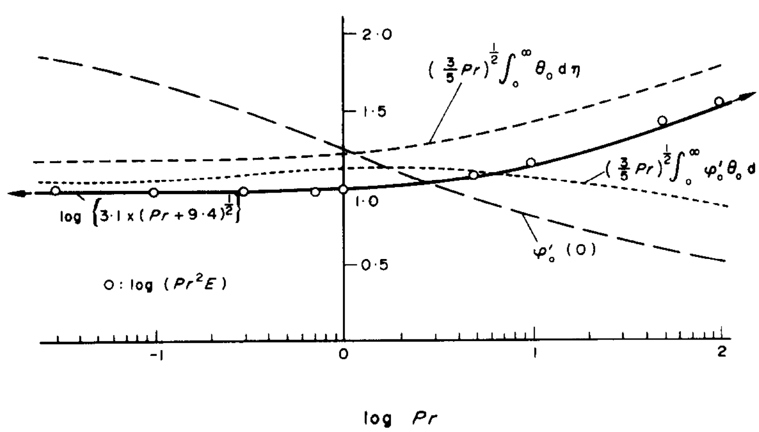
The circumferential average temperature in the far field is defined and calculated as follows

$$\left.\begin{aligned} f &\equiv \frac{1}{\pi r}\int_0^\infty t \, dx = A_0 \frac{1}{Gr}(NuGr)^{3/5} y^{-1/5} r^{-1}, \\ A_0 &\equiv \frac{1}{\pi}\left(\frac{1}{\pi} Pr \int_0^\infty \varphi_0'\theta_0 \, d\eta\right)^{-3/5}\left(\int_0^\infty \theta_0 \, d\eta\right), \end{aligned}\right\} \qquad (2.19)$$

where $r$ and $y$ must be regarded as the same value.

(iii) *Joining of two fields*

The solutions in the two fields have an as yet undetermined value of the Nusselt number $Nu$ in common. They must be connected in a suitable way




(b)

FIG. 2. Calculated results of $v$-velocity $\varphi_0'$, temperature $\theta_0$ and $E$ for pure free convection.

to give the heat-transfer coefficient. It appears too difficult to match the two solutions exactly. Hence, we take a crude form of matching that the circumferential average temperatures in the two fields may be joined up smoothly at a certain distance $r = r_j (\gg 1)$ on the $y$-axis, where $r_j \equiv y_j$. The two kinds of the integration in the exact solution become nearly equal when the width of the plume becomes small compared with the distance from the cylinder. The necessary and sufficient conditions that equations (2.10) and (2.19) should join up smoothly at the point $r_j$ are

$$\bar{t} = \hat{t}, \quad \frac{d\bar{t}}{dr} = \frac{d\hat{t}}{dr}, \quad \frac{d^2\bar{t}}{dr^2} = \frac{d^2\hat{t}}{dr^2}.$$

The heat fluxes in the two fields are to be joined up in consequence of the heat integration in the boundary condition (2.15). From the above three equations, the three constants $r_j$, $D$ and $Nu$ are determined by use of the approximation that $r \doteqdot r-1 (\gg 1)$, as follows.

$$r_j = C_A(NuGr)^{-1/3}, \quad C_A \equiv (\tfrac{42}{25}A_0)^{5/6}, \quad (2.20)$$

$$D = C_B \frac{1}{Gr}(NuGr)^2, \quad C_B \equiv \tfrac{12}{25}A_0(\tfrac{42}{25}A_0)^{-7/2}, \quad (2.21)$$

$$\frac{1}{Nu} = \tfrac{1}{3}\ln E - \tfrac{1}{3}\ln(NuGr),$$
$$E \equiv C_A^3 \exp(\tfrac{75}{42} - C_A^3 C_B). \quad (2.22)$$

The calculated results and the values of $E$ are shown against $\log Pr$ in Fig. 2(b) (for the case of the extreme Prandtl numbers, see Appendix). The values of $E$ show slightly a minimum in the vicinity of $Pr = 0{\cdot}72$. The approximation to $E$ can now be written simply as

$$E = 3{\cdot}1(Pr+9{\cdot}4)^{1/2}Pr^{-2}, \quad (2.23)$$

being presented by the solid line in the figure. This approximation is applicable in the whole range of $Pr$ including the extreme cases of $Pr = 0, \infty$, although it has small discrepancy near $Pr = 50$.

The joining distance $r_j$ may be regarded to correspond to the thickness of a concentric cylindrical film of the Langmuir concept and will give a restriction on the range of Grashof number for the present result. If the upward laminar flow is presumed to be critical at $r_{\text{crit}} = (Re_{\text{crit}} Nu^{-2/5} Gr^{-2/5})^{5/6}$ where the Reynolds number based on $r$ becomes equal to the critical one $Re_{\text{crit}} (\gg 1)$, the following relation is obtained from equation (2.20).

$$\frac{r_{\text{crit}}}{r_j} \sim Re_{\text{crit}}^{5/6} \gg 1.$$

Therefore, the far-field solution will be valid for the above limited region which is slightly different from the value given by Mahony. Although the absolute value of the joining distance $r_j$ increases with decreasing the Grashof number, the ratio of this distance to the width of the plume at the distance is invariable because they are the similar functions of $Gr$.

Similarly, the necessary and sufficient conditions for joining equations (2.11) and (2.19) are

$$\bar{t} = \hat{t}, \quad \frac{d\bar{t}}{dr} = \frac{d\hat{t}}{dr}.$$

Then, the two unknowns are approximately obtained as follows

$$r_j = C'_A(NuGr)^{-1/3}, \quad C'_A \equiv (\tfrac{6}{5}A_0)^{5/6}, \quad (2.24)$$

$$\frac{1}{Nu} = \tfrac{1}{3}\ln E' - \tfrac{1}{3}\ln(NuGr), \quad E' \equiv (\tfrac{6}{5}A_0 e)^{5/2}. \quad (2.25)$$

Although equation (2.25) is in the similar form as equation (2.22), the former gives slightly smaller Nusselt number than the latter because of the relation $E' \approx 1{\cdot}17 \times E$. The difference is only about 2 per cent for $Nu$ at $Gr = 10^{-3}$ and decreases rapidly with decreasing Grashof number. It is found that equation (2.22) is in better agreement with the experiments by Collis and Williams [5] than equation (2.25) within the limits of Fig. 3.

(iv) *Comparison with experiments*

The result of equation (2.22) with (2.23) is compared with the experimental correlations proposed by Collis and Williams [5] and by Tsubouchi, Sato and Nagakura [6] in Fig. 3. Although equation (2.22) is likely to give the slightly different slope from the

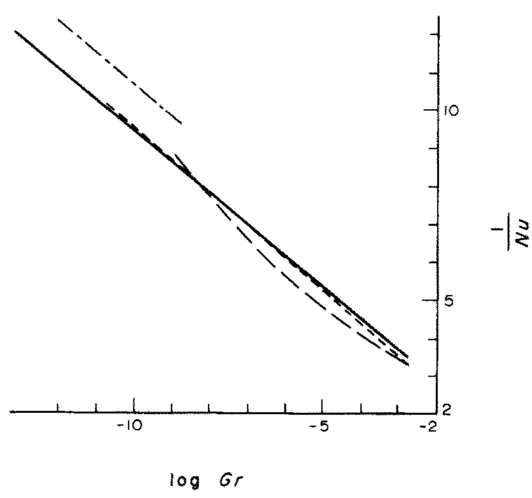



FIG. 3. Comparison of equation (2.22) and proposed correlations for pure free convection. ——, equation (2.22) for $Pr = 0{\cdot}72$; ----, Collis and Williams, $2Nu_\infty = 1/\{0{\cdot}88 - 0{\cdot}43\log(8Gr_\infty)\}$, $10^{-10} < 8Gr_\infty < 10^{-2}$ for air; —–, Tsubouchi, Sato and Nagakura, $Nu_m = 0{\cdot}466(PrGr_x)^{1/15}$ $10^{-8} < 8Gr_\infty < 10^{-2}$ for $Pr = 0{\cdot}72$; —·—·—, Mahony, for $Pr = 1$.

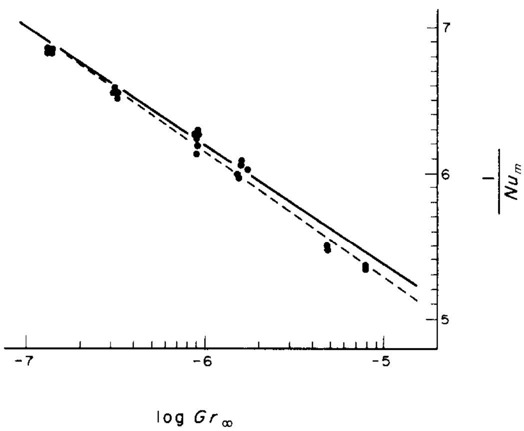



FIG. 4. Experimental results for pure free convection. ●, The authors, $20\,000 \leqslant l/a \leqslant 25\,000$; ----, Collis and Williams (for legend see Fig. 3). (——, equation (2.22) for $Pr = 0{\cdot}72$.)

correlation of Collis and Williams, the curves given by the three relations are considerably closely positioned and especially the agreement between equation (2.22) and the correlation of Collis and Williams is remarkably good. It is expected that the slope of equation (2.22) becomes more appropriate at very small Grashof numbers. The result calculated with the method proposed by Mahony gives the parallel line to equation (2.22) as shown in Fig. 3 because of the same dependency of $Gr$ on $Nu$ as in equation (2.22), ($E_{\text{Mahony}} \approx 60 \times E$ for $Pr = 1$).

The experimental result made by the authors (in detail, see Part II) with wires of length/radius ratio of 20000 to 25000 placed in air is plotted in Fig. 4. The agreement among these relations is also extraordinarily good within the experimental scattering and will reveal that the experiment by Collis and Williams is also reliable in the range of $Gr$ given in the legend of Fig. 3.

The comparison of the present result with experiments by Gebhart and Pera [8] is shown in Fig. 5 for $Pr = 6{\cdot}3$ and 63 and for the relatively large values of $Gr$. At $Pr = 50$, the value of $1/Nu$ given by equation (2.22) with the computed value of $E$ is almost 0·1 larger than that by equation (2.22) with (2.23). Therefore, it may be inferable that the agreement between equation (2.22) with the computed value of $E$ and the experiments becomes satisfactory even at $Pr = 63$.

The validity of equation (2.22) requires $r_j \gg 1$ and this produces a possible limitation of the present result. Eliminating $Nu$ from equation (2.20) by use of equations (2.22) and (2.23), we can express $r_j$ as a monotonic function of $Pr^2(Pr+9{\cdot}4)^{-1/2}Gr$, and obtain the criterion

$$Pr^2(Pr+9{\cdot}4)^{-1/2}Gr \ll 1.$$

By virtue of good agreement shown in Fig. 3 in the range of $Gr \leqslant 10^{-3}$, the above criterion can be rewritten in a further simplified form as

$$Pr^2 Gr \leqslant 10^{-3}, \qquad (2.26)$$

which implies $r_j \geqslant 18$. Even at $Pr^2 Gr = 10^{-3}$, the difference from the experimental correlation of Collis and Williams is at most 2 per cent for $Nu$. It is noted that the extended curve of equation (2.22) in Fig. 3 for $Pr = 0{\cdot}72$ passes through the point ($\log Gr = 0$, $1/Nu = 1$), takes the maximum value of $Gr$ at the point ($\log Gr = 0{\cdot}368$, $1/Nu = 0{\cdot}333$) and then goes back to the minus infinity of $\log Gr$ along the line of $1/Nu = 0$ as an asymptote ($\log Gr \to -\infty$, $1/Nu = 0$). Therefore, it seems that the criterion of the applicable range of $Gr$ can be extended up to $Gr = 10^{-2}$ or $10^{-1}$ (the

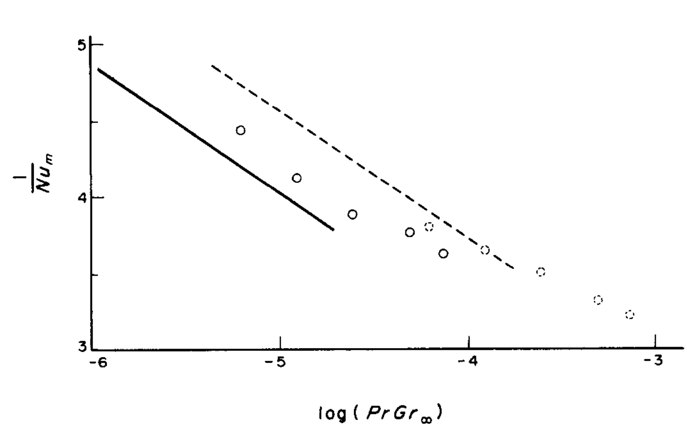



FIG. 5. Comparison of equation (2.22) with (2.23) and experimental result at $l/a = 32\,000$ by Gebhart and Pera for pure free convection. ----, ○, $Pr = 6{\cdot}3$; ——, ○, $Pr = 63$.

corresponding joining distance $r_j = 8{\cdot}3$ or $3{\cdot}9$, respectively) if the error in the Nusselt number is more allowable, because the curve of equation (2.22) is positioned over that of the experimental correlation of Collis and Williams at least near $Gr = 10^{-3}$ in the figure.

### 3. PURE FORCED CONVECTION

Next, we consider the steady uniform flow in the $x$-axis direction past a horizontal circular wire. Denoting the coordinates, velocity components and temperature as $(ax, ay)$, $(U_0 u, U_0 v)$ and $T_\infty + (T_w - T_\infty)t$ respectively, we obtain the dimensionless fundamental equations governed by Reynolds number $Re$,

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \quad (3.1)$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \quad (3.2)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (3.3)$$

$$u\frac{\partial t}{\partial x} + v\frac{\partial t}{\partial y} = \frac{1}{PrRe}\left(\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2}\right). \quad (3.4)$$

A perturbation procedure in terms of the small parameter similar to that in the foregoing section can be taken for small values of Reynolds number.

(i) *The near field*

The solutions of the governing equations in the near field can be also obtained in the same form as for the case of pure free convection although they have different units of the non-dimensionalization, and then equations (2.10) and (2.11) can be adopted for these solutions.

(ii) *The far field*

The fact that the width of the wake at a large distance from the cylinder is sufficiently small compared with the distance allows the similarity solution for the far field which may hold even at small Reynolds numbers. Then, $\Phi$, $t$ and $p$ will be expanded in the power series of $x^{-1/2} (\ll 1)$ with a similarity variable $\xi = Z_R x^{-1/2} y$ as

$$\Phi = x^{1/2}\{F_0 . f_0(\xi) + x^{-1/2} F_1 . f_1(\xi) + (x^{-1/2})^2 F_2 . f_2(\xi) + \ldots\}, \quad (3.5)$$

$$t = x^{-1/2}\{G_0 . g_0(\xi) + x^{-1/2} G_1 . g_1(\xi) + (x^{-1/2})^2 G_2 . g_2(\xi) + \ldots\}, \quad (3.6)$$

$$p = x^{-1}\{H_0 . h_0(\xi) + x^{-1/2} H_1 . h_1(\xi) + (x^{-1/2})^2 H_2 . h_2(\xi) + \ldots\}, \quad (3.7)$$

where $\Phi$ is the stream function for the forced convection case. The boundary conditions are transformed to

$$\left.\begin{aligned} &y = 0: \quad \frac{\partial \Phi}{\partial x} = \frac{\partial^2 \Phi}{\partial y^2} = \frac{\partial t}{\partial y} = 0, \quad \Phi = 0; \\ &y \to \infty: \quad \frac{\partial \Phi}{\partial y} \to 1, \quad t \to 0; \\ &\int_0^\infty \left(\frac{\partial \Phi}{\partial y} t - \frac{1}{PrRe}\frac{\partial t}{\partial x}\right) \mathrm{d}y = \pi \frac{Nu}{PrRe}; \\ &\int_0^\infty \left\{\frac{\partial \Phi}{\partial y}\left(1 - \frac{\partial \Phi}{\partial y}\right) - p\right\} \mathrm{d}y = \tfrac{1}{2} \times D_{\mathrm{drag}}(Re), \end{aligned}\right\} \quad (3.8)$$

where the last equation is derived from the drag relation of a cylinder in a uniform flow.

In the similar manner to the case of pure free convection, the governing equations and boundary conditions for the zeroth-order approximation are

$$-f_0 f_0'' = 2 f_0''', \quad (3.9)$$

$$f_0' g_0 + f_0 g_0' = \frac{-2}{Pr} g_0'', \quad (3.10)$$

$$\left.\begin{aligned} &\xi = 0: \quad f_0 = f_0'' = t' = 0; \\ &\xi \to \infty: \quad f_0' \to 1, \quad t \to 0; \\ &\int_0^\infty f_0' g_0 \,\mathrm{d}\xi = \pi \frac{Nu}{PrRe^{0{\cdot}5}} \frac{1}{G_0}; \\ &\int_0^\infty \{f_0'(1 - f_0')\} \,\mathrm{d}\xi = 0, \end{aligned}\right\} \quad (3.11)$$

where $Z_R = Re^{1/2}$ and $F_0 = Re^{-1/2}$, whereas the value of $G_0$ need not be determined. Because the integral of the drag relation is the order of magnitude of $x^{1/2}$, the right hand of the equation must be equal to zero, and the drag condition will take part in to the higher-order approximations. The solutions of equations (3.9) and (3.10) become

$$f_0 = \xi, \quad (\equiv \phi_0), \quad (3.12)$$

$$g_0 = G_{000} \exp(-\tfrac{1}{4} Pr \xi^2), \quad (\equiv G_{000} \chi_0), \quad (3.13)$$

where $G_{000} \equiv g_0(0) = (\sqrt{\pi}) Pr^{-1/2} Re^{-1/2} Nu G_0^{-1}$. These solutions illustrated in Fig. 6 show that there exist a thin thermal wake in the uniform flow.

Consequently, the circumferential average temperature is obtained as

$$\bar{t} \equiv \frac{1}{\pi r} \int_0^\infty t \,\mathrm{d}y = \frac{Nu}{PrRe} r^{-1}. \quad (3.14)$$

(iii) *Joining of two fields*

Equations (2.10) and (3.14) are to be joined smoothly at $r = r_j$ with the approximation that $r \doteqdot r - 1 (\gg 1)$, in the same manner as equations (2.11) and (2.19) for the case of pure free convection. From the three relations equating the average temperatures and their

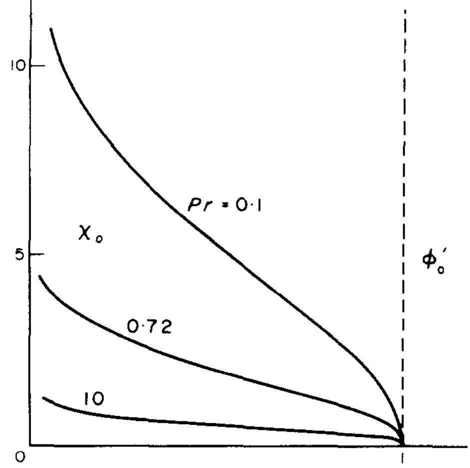



FIG. 6. Calculated results of $u$-velocity $\phi'_0$ and temperature $\chi_0$ for pure forced convection.

derivatives, the three unknowns $r_j$, $D$ and $Nu$ are obtained as follows

$$r_j = \tfrac{4}{3}(PrRe)^{-1}, \qquad (3.15)$$

$$D = \tfrac{27}{256}(PrRe)^3, \qquad (3.16)$$

$$\frac{1}{Nu} = \tfrac{2}{3} + \ln\tfrac{4}{3} - \ln(PrRe), \qquad (3.17)$$

where $\frac{2}{3} + \ln(\frac{4}{3}) \sim 0{\cdot}955$. When equations (2.11) and (3.14) are joined with the two relations, the following results are obtained.

$$r_j = (PrRe)^{-1}, \qquad (3.18)$$

$$\frac{1}{Nu} = 1 - \ln(PrRe). \qquad (3.19)$$

The value of $1/Nu$ given by equation (3.19) is only about 1·5 per cent larger than that by equation (3.17) at $PrRe = 10^{-1}$ and the difference of these values decreases rapidly with $Re$. Equation (3.17) will, however, give certainly a better approximation at very small Reynolds numbers. If we take $(r-1)^3 \doteqdot r^3 - 3r^2$ instead of $(r-1)^3 \doteqdot r^3$, the term of $\frac{3}{32}PrRe$ will be added to the r.h.s. of equation (3.17) to give the slope-variation more resemblant to that of Hieber and Gebhart [2] in the narrow range near $PrRe = 10^{-1}$. However, equation (3.19) is simpler and in slightly better agreement with the experiments by Collis and Williams [7] for Reynolds numbers in the range of Fig. 7 than equation (3.17).

(iv) *Comparison with experiments*

The simple relation of equation (3.19) is similar to the result of Cole and Roshko [2] with Oseen's approximation and agrees well with the correlation proposed by Collis and Williams [7] as shown in Fig. 7, although the complicated relation analyzed in detail by Hieber and Gebhart [2] is in better agreement with the current result.

The experimental result made by the authors with wires of the aspect ratio of length/radius of 20000 to 25000 is plotted in Fig. 8 (in detail, see Part II), and compared with the above theory. It is seen from the figure that the agreement among the three is considerably good. The effects of the aspect ratio and the temperature loading are sufficiently negligible within the measurement errors of the present experiments.

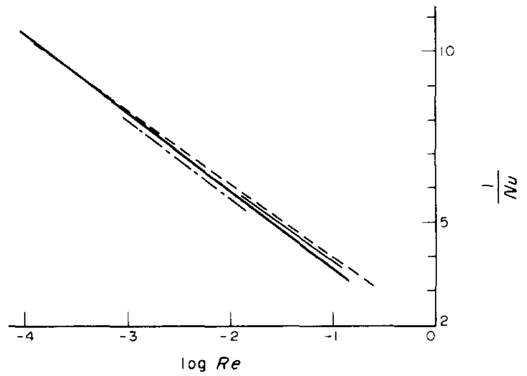



FIG. 7. Comparison of equation (3.19) and proposed correlations for pure forced convection. ——, equation (3·19) for $Pr = 0{\cdot}72$; ----, Collis and Williams, $2Nu_m(T_m/T_\infty)^{-0{\cdot}17} = 1/\{1{\cdot}18 - 1{\cdot}10\log(2Re_m)\}$, $0{\cdot}5 > 2Re_m(> 0{\cdot}01)$ for air, assuming $T_m/T_\infty = 1$; ·—, Hieber and Gebhart, $2Nu = (2/\alpha)(1 - a/\alpha^2)$; $\alpha = \ln(4/\gamma PrRe)$; $a = 1{\cdot}38$ for $Pr = 0{\cdot}7$; —·—, Cole and Roshko, $1/Nu = \ln(4/PrRe) - \gamma_E$ for $Pr = 0{\cdot}72$.

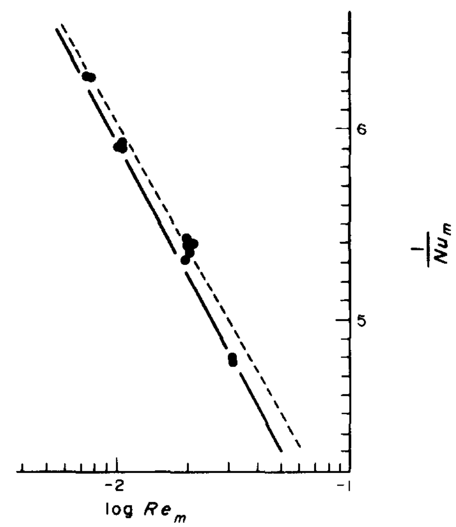



FIG. 8. Experimental results for pure forced convection. ●, The authors, $20000 \leqslant l/a \leqslant 25000$; ----, Collis and Williams (for legend see Fig. 7). (——, equation (3.19) for $Pr = 0{\cdot}72$.)

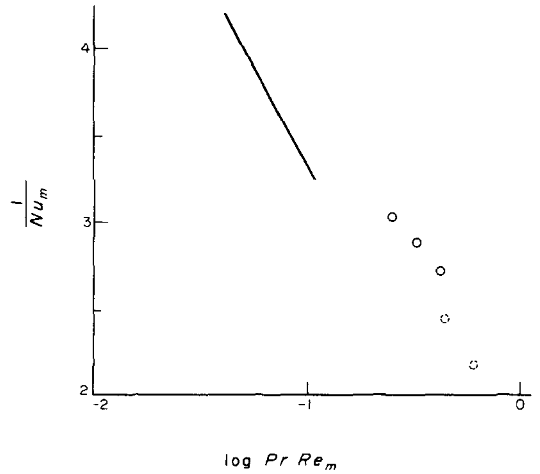



FIG. 9. Comparison of equation (3.19) and experimental result at $l/a = 32000$ by Gebhart and Pera for pure forced convection. ◌, $Pr = 6{\cdot}3$; ○, $Pr = 63$.

The deviations of the experimental points from the mean value, which are grouped at constant Reynolds number, excluding the scattering of particular points, are found to be mainly caused by the measurement error of the diameter of the fine wire because they consist almost of four groups as seen commonly in Figs. 4 and 8, that is, 0·00064 cm dia ($Gr_\infty = 1{\cdot}30 \times 10^{-7}$; $Re_m = 7{\cdot}6 \times 10^{-3}$), 0·00088 cm dia ($Gr_\infty = 3{\cdot}2 \times 10^{-7}$; $Re_m = 9{\cdot}3 \times 10^{-3}$), 0·00164 cm dia ($Gr_\infty = 0{\cdot}91$ and $1{\cdot}51 \times 10^{-6}$; $Re_m = 2{\cdot}05 \times 10^{-2}$) and 0·00262 cm dia ($Gr_\infty = 4{\cdot}8$ and $8{\cdot}0 \times 10^{-6}$; $Re_m = 3{\cdot}10 \times 10^{-2}$). Moreover, the deviated amounts of Nusselt number for each diameter are roughly equal for both cases of pure free convection and pure forced convection. This may be concluded from the fact that Nusselt numbers given by equations (2.22) and (3.19) are of the similar function of the diameter.

Comparison with the experiments by Gebhart and Pera [8] for $Pr = 6{\cdot}3$ and 63 is shown in Fig. 9, although the Reynolds numbers of their results exceed the applicable limit of equation (3.19). Nevertheless, it is evident from the figure that equation (3.19) appears to be valid approximately up to such higher Prandtl numbers.

In the same manner as the case of pure free convection, the applicable range of equation (3.19) can be derived from the restriction $r_j \gg 1$ and the comparison with the experimental results in Fig. 7 to be expressed as

$$PrRe \leqslant 10^{-1}, \tag{3.20}$$

which implies $r_j \geqslant 10$. The difference between the present result and the correlation of Collis and Williams are about 8 and 3 per cent at $PrRe = 10^{-1}$ and $10^{-2}$ respectively, although they decrease rapidly with the decrease of $PrRe$.

## 4. CONCLUSION

By analyzing the heat-transfer fields from a circular wire by pure free convection at small Grashof numbers or pure forced convection at small Reynolds numbers and by using the method of joining the circumferential average temperatures, the correlations between Nusselt number and Grashof number or Reynolds number are obtained to be in good agreement with the experimental results. The derived correlations of heat transfer are

$$\frac{1}{Nu} = \tfrac{1}{3}\ln E - \tfrac{1}{3}\ln(NuGr),$$
$$E = 3{\cdot}1(Pr+9{\cdot}4)^{1/2}Pr^{-2}, \quad Pr^2Gr \leqslant 10^{-3};$$
$$\frac{1}{Nu} = 1 - \ln(PrRe), \quad PrRe \leqslant 10^{-1},$$

for cases of pure free convection and pure forced convection, respectively. The applicable range of $Pr$ for the present results is not to be restricted, excluding for the case of rarefied gases.

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## APPENDIX

When the coefficients $Z_G$ and $\pi_0$ are taken to be $Z_G = (\frac{9}{25}Pr^2Gr\Theta_0)^{1/4}$ and $\pi_0 = (\frac{25}{9}Pr^{-2}Gr\Theta_0)^{1/4}$ as Mahony [4] does, equations (2.16), (2.17) and (2.18) become,

$$\varphi_{0m}'^2 - 3\varphi_{0m}\varphi_{0m}'' = 5\theta_{0m} - 3Pr\varphi_{0m}''', \tag{1a}$$

$$\theta_{0m}' = \theta_{0m}\varphi_{0m}, \tag{2a}$$

$$\left.\begin{aligned} &\eta_{0m} = 0: \quad \varphi_{0m} = \varphi_{0m}'' = 0, \quad \theta_{0m} = 1; \\ &\eta_{0m} \to \infty: \quad \varphi_{0m}' \to 0; \\ &\Theta_0 = \left\{\frac{1}{\pi}(\tfrac{5}{3}Pr)^{1/2}Gr^{1/4}Nu^{-1}\int_0^\infty \varphi_{0m}'\theta_{0m}\,d\eta_{0m}\right\}^{-4/5}. \end{aligned}\right\} \tag{3a}$$

The numerical results of the above equations are shown in Fig. 10 for $Pr = 0{\cdot}01$ and 1000.

It may be estimated from the calculated results that the term $Pr\varphi_{0m}'''$ in equation (1a) could be omitted in the whole range of $\eta_{0m}$ as $Pr \to 0$. Therefore, at the limit of $Pr = 0$, equations (1a) and (2a) may be solved without the term of

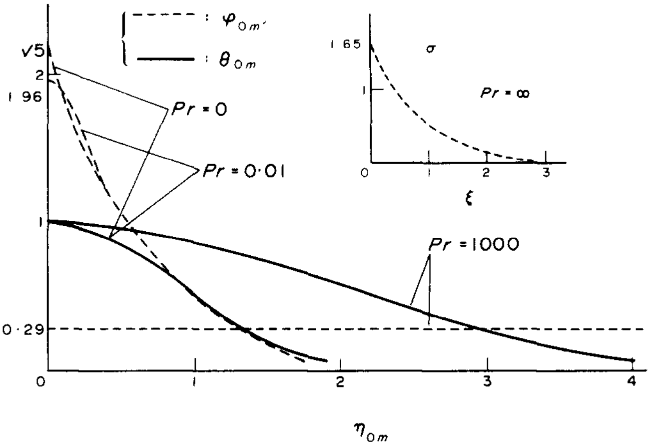


FIG. 10. Distributions of $v$-velocity $\varphi'_{0m}$ and temperature $\theta_{0m}$ at extreme Prandtl numbers for pure free convection.

$Pr\varphi'''_{0m}$ subject to the boundary conditions

$$\left.\begin{array}{l}\eta_{0m}=0:\quad \varphi_{0m}=0,\qquad \theta_{0m}=1;\\ \eta_{0m}\to\infty:\ \varphi'_{0m}\to 0.\end{array}\right\}\tag{4a}$$

The numerical results calculated with the Runge–Kutta–Gill method give

$$\left.\begin{array}{r}\varphi'_{0m}(0)=2{\cdot}187,\\ \displaystyle\int_0^\infty \varphi'_{0m}\theta_{0m}\,d\eta_{0m}=1{\cdot}035,\\ \displaystyle\int_0^\infty \theta_{0m}\,d\eta_{0m}=1{\cdot}194.\end{array}\right\}\tag{5a}$$

In this calculation, the value of $\varphi''_{0m}$ is discontinuous at $\eta_{0m}=0$. Here, if $\varphi'_{0m}$ near $\eta_{0m}=0$ can be given by

$$\varphi'_{0m}=(\sqrt{5})\left\{1+a_1\eta_{0m}^{2/3}+\frac{a_2}{2!}(\eta_{0m}^{2/3})^2+\ldots\right\},\tag{6a}$$

the exact values of $\varphi'_{0m}$ near $\eta_{0m}=0$ will be obtained by evaluating the parameters $a_1$, $a_2$, ... as it joins smoothly to the computed values of $\varphi'_{0m}$. The exact solution is also shown in Fig. 10. However, at the limit of $Pr=0$, the value of $E$ can be approximately derived only from the numerical integration as follows

$$E=9{\cdot}96Pr^{-2}.\tag{7a}$$

At extremely large Prandtl numbers, the wake width of the velocity $\varphi'_{0m}$ becomes large compared with that of temperature $\theta_{0m}$ as shown in Fig. 10. Then, if $\eta_{0m}=\eta_{0m0}$ is the position where $\theta_{0m}$ becomes almost zero, the approximation $\varphi_{0m}=\varphi'_{0m}(0)\,.\,\eta_{0m}$ may be assumed in the range of $0\leqslant\eta_{0m}\leqslant\eta_{0m0}$. Equation (2a) is then solved as follows

$$\theta_{0m}=\exp\{\tfrac{1}{2}\varphi'_{0m}(0)\,.\,\eta_{0m}^2\}.\tag{8a}$$

And also equation (1a) is integrated from zero to $\eta_{0m}(>\eta_{0m0})$,

$$4\int_0^{\eta_{0m}}(\varphi'_{0m})^2\,d\eta_{0m}-3\varphi'_{0m}\int_0^{\eta_{0m}}\varphi'_{0m}\,d\eta_{0m}=\frac{5\sqrt{\pi}}{2}\left\{\frac{-\varphi'_{0m}(0)}{2}\right\}^{-1/2}-3Pr[\varphi''_{0m}]_0^{\eta_{0m}}.\tag{9a}$$

With the relations of $\eta_{0m}=Pr^{5/8}\zeta(\eta_{0m0}=Pr^{5/8}\zeta_0)$ and $\varphi'_{0m}=Pr^{-1/4}\sigma(\varphi'_{0m}(0)=Pr^{-1/4}\sigma_0)$, the above equation and its boundary conditions may be reduced to

$$4\int_0^\zeta\sigma^2\,d\zeta-3\sigma\int_0^\zeta\sigma\,d\zeta=\frac{5\sqrt{\pi}}{2}\left(\frac{-\sigma_0}{2}\right)^{-1/2}-3\frac{d\sigma}{d\zeta},\tag{10a}$$

$$\left.\begin{array}{l}\zeta=0:\quad \dfrac{\partial\sigma}{\partial\zeta}=\dfrac{5\sqrt{\pi}}{6}\left(\dfrac{-\sigma_0}{2}\right)^{-1/2};\\ \zeta\to\infty:\ \sigma\to 0.\end{array}\right\}\tag{11a}$$

which is of an initial-value problem with respect to $\sigma_0$. The numerical result of $\sigma$ is shown in Fig. 10.

$$\sigma_0=-1{\cdot}652.\tag{12a}$$

Consequently, at extremely large Prandtl numbers, with the approximation

$$\int_0^\infty\varphi'_{0m}\theta_{0m}\,d\eta_{0m}=\varphi'_{0m}(0)\int_0^\infty\theta_{0m}\,d\eta_{0m},$$

the following result is obtained.

$$E=3{\cdot}10Pr^{-3/2}\tag{13a}$$

The approximation (2.23) is determined so as to satisfy these extreme cases strictly besides its behavior around $Pr=1$.